

Tensor-Scalar Torsion

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Abstract

A theory of gravity with torsion is examined in which the torsion tensor is constructed from the exterior derivative of an antisymmetric rank two potential plus the dual of the gradient of a scalar field. Field equations for the theory are derived by demanding that the action be stationary under variations with respect to the metric, the antisymmetric potential, and the scalar field. A material action is introduced and the equations of motion are derived. The correct conservation law for rotational angular momentum plus spin is observed to hold in this theory.

1 Introduction

Recently, a theory of gravitation with torsion of the form¹

$$S_{\mu\nu\sigma} = \psi_{[\mu\nu,\sigma]} , \quad (1)$$

where $\psi_{\mu\nu}$ is an antisymmetric torsion potential has been investigated [1]. There are several reasons for our interest in this approach. The theory has predictive power owing to the fact that second-order differential equations for $\psi_{\mu\nu}$ are obtained when the curvature scalar is used as the Lagrangian [2]. This theory makes definite predictions of a long range spin-spin interaction [3], helicity flip interactions [4], and radiation of torsion waves [5], although the effects may be smaller than can be measured with present technology. In this theory, the source for torsion is intrinsic spin, and it has been shown that the correct law for conservation of rotational angular momentum plus spin is consistently obtained.

One motivation for analysis of gravitational theories with torsion is the generic appearance of the antisymmetric tensor field in models which correspond to the low-energy limit of string theory. The argument that (1) may be related to string theory first requires the expansion of the curvature scalar R for U_4 spacetime² in terms of the V_4 scalar curvature oR plus torsion terms

$$R = {}^oR - S^{\alpha\beta\gamma} S_{\alpha\beta\gamma} . \quad (2)$$

This is equivalent to the low energy string theory Lagrangian, for a constant dilaton [6].

¹ Here, and in the remainder of this paper, we shall denote partial differentiation by comma and (anti)symmetrization over multiple indices by (square brackets) parentheses.

² In a U_n spacetime the metric or *fundamental tensor* is real-valued and symmetric, and the connection is metrically compatible. The additional constraint that the connection be symmetric defines a V_n spacetime. The details are to be found in [11]. Our notation is such that covariant differentiation with the Christoffel connection formed from the metric, $\{\overset{\mu}{\alpha\beta}\}$, shall be indicated by a semi-colon, while the full covariant derivative, using the entire affine connection including torsion, is expressed by the operator ∇_μ .

Another reassuring feature of this theory is that when matter sources are introduced in the usual manner, consistent physical results are obtained for phenomenological spin sources and for classical Dirac fields which possess an intrinsic spin [8]. However, in solving the torsion field equations and the Dirac equation under the *ansatz* (1), a scalar field had to be introduced as a function of integration. Denoting this additional field by χ , the solution for the torsion $S^{\mu\nu\sigma}$ in the presence of a source field Ψ which obeys the Dirac equation is

$$S^{\mu\nu\sigma} = \frac{2\pi i \hbar G}{c^3} \bar{\Psi} \gamma^{[\mu} \gamma^\nu \gamma^{\sigma]} \Psi + \epsilon^{\mu\nu\sigma\alpha} \chi_{,\alpha}. \quad (3)$$

Introducing the dual torsion b_μ , defined as $b_\mu = \epsilon_{\mu\alpha\beta\gamma} S^{\alpha\beta\gamma}$, this may be rewritten as

$$b^\mu = -\frac{12\pi \hbar G}{c^3} \bar{\Psi} \gamma^\mu \gamma_5 \Psi - 6\chi^{;\mu}. \quad (4)$$

To see that the presence of the scalar field is vital to the dynamics of the general solution, consider the following argument. From the *ansatz* (1), it follows that $b^\sigma{}_{;\sigma} = 0$ must hold identically. For the b^μ in (4), this leads to

$$\square \chi \equiv \chi^{;\mu}{}_{;\mu} = \frac{12\pi \hbar G}{c^3} (\bar{\Psi} \gamma^\mu \gamma_5 \Psi)_{;\mu}, \quad (5)$$

the right hand side of which, in general, is not zero. Therefore, the scalar field is dynamical and may not be consistently chosen to vanish.

In the present paper, we investigate the consequences of naturally incorporating a scalar field *a priori* by modifying (1). This seems reasonable on two grounds. The first is that, as shown above, a scalar field naturally arises in the solutions of this model. The second is that the low-energy limit of string theory possesses scalar modes in addition to the tensor and vector modes. Perhaps one of these modes contributes to the gravitational sector in the way that we propose. Thus, we shall assume from the start that the torsion tensor admits the decomposition

$$S_{\mu\nu\sigma} = H_{\mu\nu\sigma} + \epsilon_{\mu\nu\sigma\alpha} \lambda^{,\alpha}, \quad (6)$$

where $H_{\mu\nu\sigma} = \psi_{[\mu\nu,\sigma]}$, and λ is the scalar field.

The main result that we obtain is that no harm is done to the physically desirable law of conservation of total angular momentum (rotation plus spin). Furthermore, it is still the case in our model that spin acts as the source for the $H_{\mu\nu\sigma}$ part of the torsion, while the scalar field does not directly arise from spin.

2 Field equations

The action integral is given by

$$I = \int (\sqrt{-g}R + k\mathcal{L}_M) d^4x, \quad (7)$$

where $\sqrt{-g}R$ and \mathcal{L}_M are the gravitational and material Lagrangian densities respectively, and $k = 8\pi G/c^4$. The curvature scalar, $R = g^{\alpha\beta}R_{\alpha\beta}$, includes torsion terms. With torsion of the form (6), R expands as

$$R = {}^oR - H^{\alpha\beta\gamma}H_{\alpha\beta\gamma} + 2\epsilon^{\alpha\beta\gamma\sigma}H_{\beta\gamma\sigma}\lambda_{,\alpha} + 6\lambda^{,\alpha}\lambda_{,\alpha}. \quad (8)$$

Variations of I with respect to $g_{\mu\nu}$, $\psi_{\mu\nu}$, and λ , are assumed to vanish, yielding the classical field equations. With the following notational conventions for the sources of the metric, antisymmetric potential, and scalar fields:

$$\int \sqrt{-g}T^{(\mu\nu)}\delta g_{\mu\nu} d^4x \equiv \int \frac{\delta\mathcal{L}_M}{\delta g^{\mu\nu}}\delta g_{\mu\nu} d^4x, \quad (9)$$

$$\int \sqrt{-g}T^{[\mu\nu]}\delta\psi_{\mu\nu} d^4x \equiv \int \frac{\delta\mathcal{L}_M}{\delta\psi^{\mu\nu}}\delta\psi_{\mu\nu} d^4x, \quad (10)$$

$$\int \sqrt{-g}U\delta\lambda d^4x \equiv \int \frac{\delta\mathcal{L}_M}{\delta\lambda}\delta\lambda d^4x, \quad (11)$$

the field equations are as follows:

$${}^oG^{\mu\nu} - 3H^\mu{}_{\alpha\beta}H^{\nu\alpha\beta} + \frac{1}{2}g^{\mu\nu}H^{\alpha\beta\gamma}H_{\alpha\beta\gamma} + 6\lambda^{,\mu}\lambda^{,\nu} - 3g^{\mu\nu}\lambda^{,\alpha}\lambda_{,\alpha} = kT^{(\mu\nu)}, \quad (12)$$

$$H^{\mu\nu\sigma}{}_{;\sigma} = -\frac{k}{2}T^{[\mu\nu]}, \quad (13)$$

$$\square\lambda = \frac{U}{12}. \quad (14)$$

Note that the use of the symbol $T^{[\mu\nu]}$ to denote the source of the antisymmetric field is to allow the *formal* introduction of $T^{\mu\nu} = T^{(\mu\nu)} + T^{[\mu\nu]}$ as the source for the combined metric and torsion potential $g^{(\mu\nu)} + \psi^{[\mu\nu]}$. U describes the phenomenological source for the scalar field and oG is the V_4 Einstein tensor.

With these field equations we shall develop the equations of motion for a test body with intrinsic spin moving through a region of spacetime containing both gravitational and torsion fields. Furthermore, we verify the consistency of the conservation law for rotational plus intrinsic spin angular momentum in the case in which the external torques due to gravity and torsion vanish.

3 Equations of motion for a small test body

In reference [1] Hammond adapted the method established by Papapetrou [10] to develop the equations of motion for a small body in a gravitational and torsion field (no scalar field). This method is used here to write the equations of motion for a small body in a combined gravitational and torsion field in which the scalar field is present.

Papapetrou solved for the motion of a spinning test particle in a background gravitational field through the effective use of an iterative multipole-like expansion. There are systematic and practical advantages to this approach [10]. Spinless and structureless extended particles are zeroth-order poles, while spinning particles possess at least a pole-dipole structure. For small bodies it is possible to neglect higher-order pole terms to a reasonable approximation.

The U_4 Bianchi Identity [11] holds when torsion is present.

$$\nabla_\nu G^{\mu\nu} = 2S^{\mu\alpha\beta}R_{\beta\alpha} - S_{\alpha\beta\gamma}R^{\mu\gamma\beta\alpha}. \quad (15)$$

Substitution of the field equations and use of U_4 geometrical identities leads to

$$T^{\mu\nu}{}_{;\nu} = \frac{3}{2}H^\mu{}_{\alpha\beta}T^{\alpha\beta} + \frac{1}{6k}\lambda^{,\mu}U, \quad (16)$$

where $T^{\mu\nu} = T^{(\mu\nu)} + T^{[\mu\nu]}$, as was discussed above.

Let $\tau^{\mu\nu} = \sqrt{-g}T^{(\mu\nu)}$, $j^{\alpha\beta} = \frac{k}{K}T^{[\alpha\beta]}$, and $\tilde{j}^{\alpha\beta} = \sqrt{-g}j^{\alpha\beta}$ so that a tilde denotes density, where K is an arbitrary coupling constant for the spin field introduced by Hammond in [1]. It then follows that

$$\tau^{\mu\nu}{}_{;\nu} = \sqrt{-g}T^{(\mu\nu)}{}_{;\nu} - \{\frac{\mu}{\alpha\beta}\}\tau^{\alpha\beta}, \quad (17)$$

and therefore

$$\tau^{\mu\nu}{}_{;\nu} = \frac{3K}{2k}H^\mu{}_{\alpha\beta}\tilde{j}^{\alpha\beta} + \frac{1}{6k}\lambda^{,\mu}\tilde{U} - \{\frac{\mu}{\alpha\beta}\}\tau^{\alpha\beta}. \quad (18)$$

Consider the integral of $\tau^{\mu\nu}{}_{;\nu}$ over a spatial hypersurface characterized by fixed time. For a bounded source, the integral has support only over the finite volume of the body, and hence the surface terms which arise upon invoking the divergence theorem vanish (*viz.*, $\int \tau^{\mu i}{}_{;i} dV = 0$, where Latin indices run over spatial coordinates). Then,

$$\int \tau^{\mu 0}{}_{;0} dV = \int \left\{ \frac{3K}{2k}H^\mu{}_{\alpha\beta}\tilde{j}^{\alpha\beta} + \frac{1}{6k}\lambda^{,\mu}\tilde{U} - \{\frac{\mu}{\alpha\beta}\}\tau^{\alpha\beta} \right\} dV. \quad (19)$$

Let y^α denote the position of the center of mass of the small test body, while δx^α are the center of mass coordinates of points in the body. Thus $x^\alpha = y^\alpha + \delta x^\alpha$ describes the position of any point within the body. Expanding $\{\frac{\mu}{\alpha\beta}\}$, $H^\mu{}_{\alpha\beta}$, and $\lambda^{,\mu}$ in Taylor series about y^α to first order in the small quantities δx^α , and substituting these into (19) results in

$$\begin{aligned} \frac{d}{dt} \int \tau^{\mu 0} dV &= \int \left\{ \frac{3K}{2k}(H^\mu{}_{\alpha\beta} + H^\mu{}_{\alpha\beta,\sigma}\delta x^\sigma)\tilde{j}^{\alpha\beta} \right. \\ &\quad + \frac{1}{6k}(\lambda^{,\mu} + \lambda^{,\mu}{}_{,\sigma}\delta x^\sigma)\tilde{U} \\ &\quad \left. - (\{\frac{\mu}{\alpha\beta}\} + \{\frac{\mu}{\alpha\beta}\}{}_{,\sigma}\delta x^\sigma)\tau^{\alpha\beta} \right\} dV. \end{aligned} \quad (20)$$

The time derivative has been pulled out of the integral on the left hand side of (20), and on the right hand side it is understood that the fields and their derivatives are evaluated at the position of the center of mass.

Equation (20) is the equation of motion for the body. To see this more clearly, we note that $v^\alpha = dy^\alpha/d\tau$ is the center of mass velocity of the body (τ is the invariant proper time as measured along the worldline of the center of mass), and define the following four quantities:

$$M^{\mu\nu} \equiv \frac{v^0}{c} \int \tau^{\mu\nu} dV, \quad (21)$$

$$M^{\alpha\mu\nu} \equiv -\frac{v^0}{c} \int \delta x^\alpha \tau^{\mu\nu} dV, \quad (22)$$

$$J^{\mu\nu} \equiv \frac{1}{c} \int (\delta x^\mu \tau^{\nu 0} - \delta x^\nu \tau^{\mu 0}) dV, \quad (23)$$

$$m^{\alpha\mu\nu} \equiv \frac{v^0}{c} \int \delta x^\alpha \tilde{j}^{\mu\nu} dV. \quad (24)$$

In the language used by Papapetrou, $M^{\mu\nu}$ consists of pole terms, while $M^{\alpha\mu\nu}$ and $J^{\mu\nu}$ are dipole contributions. $m^{\alpha\mu\nu}$ is the lowest-order torsion term in this approximation scheme, and is also a dipole term. Reexpressing (20) in terms of these quantities produces

$$\begin{aligned} \frac{d}{d\tau} \frac{M^{\mu 0}}{v^0} &= \frac{3K}{2k} \frac{v^0}{c} H^\mu{}_{\alpha\beta} \int \tilde{j}^{\alpha\beta} dV + \frac{3K}{2k} H^\mu{}_{\alpha\beta, \gamma} m^{\gamma\alpha\beta} \\ &\quad - \{\alpha\beta\}^\mu M^{\alpha\beta} + \{\alpha\beta\}^\mu{}_{, \gamma} M^{\gamma\alpha\beta} \\ &\quad + \frac{1}{6} \frac{v^0}{ck} \int \lambda^{, \mu} \tilde{U} dV + \frac{1}{6} \frac{v^0}{ck} \int \delta x^\gamma \lambda^{, \mu}{}_{, \gamma} \tilde{U} dV, \end{aligned} \quad (25)$$

which, if $H^\mu{}_{\alpha\beta} = 0$ and $\lambda = 0$, reduces to the familiar set of equations for a rotating body in a gravitational field. Equation (25) is interpreted as follows, the term on the left hand side is the net force acting on the test body, and the terms on the right hand side are the forces due to the spin field and its gradient, the gravitational field and its gradient, and the scalar field and its gradient.

The equations for angular momentum and spin are obtained by integrating the divergence of the first moment of the symmetric stress-energy-momentum density,

$$(x^\sigma \tau^{\mu\nu})_{, \nu} = \tau^{\mu\sigma} + x^\sigma \frac{3K}{2k} H^\mu{}_{\alpha\beta} \tilde{j}^{\alpha\beta} - x^\sigma \{\alpha\beta\}^\mu \tau^{\alpha\beta} + \frac{1}{12k} x^\sigma \lambda^{, \mu} \tilde{U}, \quad (26)$$

over all space. The boundedness of the source yields $\int (x^\mu \tau^{\nu i})_{,i} dV = 0$, simplifying the integral of the left hand side of (26).

$$\begin{aligned} \frac{d}{dt} \int (x^\sigma \tau^{\mu 0}) dV &= \int \tau^{\mu \sigma} dV + \int x^\sigma \frac{3K}{2k} H^\mu_{\alpha\beta} \tilde{j}^{\alpha\beta} dV \\ &\quad - \int x^\sigma \{\alpha_\beta^\mu\} \tau^{\alpha\beta} dV + \int \frac{1}{12k} x^\sigma \lambda^{\cdot\mu} \tilde{U} dV. \end{aligned} \quad (27)$$

Once again, expanding $\{\alpha_\beta^\mu\}$, $H^\mu_{\alpha\beta}$, and $\lambda^{\cdot\mu}$ about y^α , and using the quantities defined above, we see that

$$\begin{aligned} M^{\mu\sigma} &= \frac{v^\sigma}{v^0} M^{\mu 0} - \frac{d}{d\tau} \frac{M^{\sigma\mu 0}}{v^0} - \frac{3K}{2k} H^\mu_{\alpha\beta} m^{\sigma\alpha\beta} \\ &\quad - \{\alpha_\beta^\mu\} M^{\sigma\alpha\beta} - \frac{v^0}{12kc} \lambda^{\cdot\mu} \int \delta x^\sigma \tilde{U} dV. \end{aligned} \quad (28)$$

Following upon its construction, $M^{\mu\sigma}$ must be symmetric in its indices, which requires, after some simplification, that

$$\begin{aligned} \frac{d}{d\tau} J^{\sigma\mu} + v^\sigma \frac{M^{\mu 0}}{v^0} - v^\mu \frac{M^{\sigma 0}}{v^0} &= \frac{3K}{k} H^{[\mu}_{\alpha\beta} m^{\sigma]\alpha\beta} + 2\{\alpha_\beta^{[\mu}\} M^{\sigma]\alpha\beta} \\ &\quad + \frac{v^0}{kc} \lambda^{\cdot[\mu} \int \delta x^{\sigma]} \tilde{U} dV. \end{aligned} \quad (29)$$

The right hand side of (29) is the net applied torque on the body due to the spin field, the gravitational field, and the scalar field. The terms on the left side represent the proper time derivatives of the rotational angular momentum plus spin, and the orbital angular momentum.

As defined in (23), $J^{\sigma\mu}$ is the total angular momentum. In Hammond's original model and again in the present case it is composed of $L^{\sigma\mu}$, the rotational angular momentum, plus $S^{\sigma\mu}$, arising from intrinsic spin which is not associated with any physical motion of the body:

$$J^{\sigma\mu} \equiv L^{\sigma\mu} + S^{\sigma\mu}. \quad (30)$$

When the applied net torque from all sources vanishes, (29) may be integrated along the worldline of the test body to yield an expression of the law of conservation of angular momentum

$$L^{\sigma\mu} + S^{\sigma\mu} + y^\sigma p^\mu - p^\sigma y^\mu = \text{constant}. \quad (31)$$

4 Conclusion

The existence of rather general arguments in favor of the addition of a scalar field to the torsion sector of Hammond's theory was in no way a guarantee that such an augmented model would remain self-consistent. In this paper we have shown that the inclusion of the scalar field in the specified manner with a classical phenomenological source may be achieved without a breakdown in the conservation laws governing the motion of classical test bodies. This result suggests that the model deserves further investigation.

The only serious deficiency that we have noted is that the source of the scalar field is nearly unconstrained classically – unlike the case of the $\psi_{[\mu\nu]}$ torsion potential in Hammond's original model. Now that the augmented model has passed its first hurdle, we plan to incorporate a classical Dirac field as the source of stress-energy and spin. It is hoped that this analysis may shed light upon the physical origin and properties of the additional scalar field.

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